Variable-Property Effects in Supersonic Wedge Flow

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Abstract

A STUDY is presented on the variable-property effects on supersonic flow along a wedge, taking into account wall thermal resistance. The axis of the wedge is maintained either at a constant temperature or under adiabatic conditions. The equation of heat conduction inside the wall is transformed by a suitable approximation into boundary conditions for the equations of heat transfer in the fluid. Viscosity and thermal conductivity coefficients are assumed to depend on temperature in a polynomial form. The problem governed by the isothermal condition on the axis does not admit similarity solutions and is solved by using two expansions, an initial one and an asymptotic one, and applying the Padé approximants technique. The solution of the second problem can be obtained in similarity form. A discussion of the variable-property effects and the influence of thermal resistance of the wall on the Nusselt number friction coefficient, and temperature at the wall, for several values of the Mach number, ends the paper.

Contents

General

The first studies on variable-property effects were based on empiric methods, such as the "reference-temperature method" and the "property-ratio method" (see, e.g., Kays[1]). In the reference-temperature method, the properties are calculated at a reference temperature T_r different for each property. In the property-ratio method, the variable-property results are obtained by multiplying the corresponding constant-property results by a factor in the form $\Pi[\alpha_i(T_w)/\alpha_i(T_\infty)]^{n_i}$, where $\alpha_1 = \mu$, $\alpha_2 = \lambda$, $\alpha_3 = \rho$ and the exponents n_i are determined empirically. Recently, the problem of variable-property effects has been studied, for walls of zero thickness, in the case of small temperature differences. In particular, Carey and Mollendorf[2] have presented a first-order perturbation analysis for liquids assuming a linear dependence of viscosity on temperature. Gray and Giorgini[3] have analyzed the limits of applicability of the Boussinesq approximation in the natural convection. Merker and Mey,[4] studying the natural convection in a shallow cavity, have found that the reference temperature can be assumed to be the arithmetic mean between the highest and lowest temperature, if the difference of these temperatures is on the order of 30 K or less.

The temperature field in the solid is governed by the steady heat-conduction equation that in polar coordinates ϑ and r is $T_{\vartheta\vartheta} + (rT_r)_r = 0$. Let us now study this equation in a wedge of half-angle α, assuming that the surface at $\vartheta = 0$ is maintained 1) at a constant temperature T_a (this case is of interest for refrigerated bodies) and 2) under adiabatic conditions. In the first case, if R is a characteristic length in the r direction, it is convenient to introduce the dimensionless coordinates $r' = r/R$ and $\vartheta' = \vartheta/\alpha$. Then, the heat-conduction equation may be written as $T_{\vartheta'\vartheta'} = -\alpha^2 r'(r'T_{r'})_{r'}$ and the temperature may be expanded in a Maclaurin series in powers of α^2. The leading term of this expansion, governed by the equation $T_{\vartheta'\vartheta'} = 0$, is given by $T = T_a + (T_w - T_a)\vartheta/\alpha$ and, hence, the normal derivative at the upper surface is given by $T_{n,w} = (T_w - T_a)/\alpha r$. In the second case, $T_{\vartheta'}$ may be expanded in a Maclaurin series in terms of ϑ'. From the heat-conduction equations, one has for the leading term $T_{n,w} = -\alpha(rT_r)_{r,w'}$, which is the boundary condition to be used for the flow equations in the case of adiabatic condition on the axis. To describe the thermofluiddynamic field downstream of the shock wave, one must solve the coupled thermal fields in the solid and in the fluid, giving a continuity condition both for temperature and heat flux at the solid-fluid interface.

The equations governing the thermofluiddynamic field are the boundary-layer equations, which may be written in nondimensional form as

$$(\rho u)_x + (\rho v)_y = 0 \tag{1a}$$

$$\rho(uu_x + vu_y) = (\mu u_y)_y \tag{1b}$$

$$\rho(ut_x + vt_y) = (\lambda t_y)_y/Pr + [(\gamma - 1)M_2^2/\Delta t]\mu u_y^2 \tag{1c}$$

where t is the dimensionless temperature defined as $(T - T_2)/(T_a - T_2)$ in the first case and $(T - T_2)/T_2$ in the second case, and the symbols have their usual meaning. The boundary conditions associated with Eqs. (1) are $u(x,0) = v(x,0) = 0$, $u(x,\infty) = 1$, $t(x,\infty) = 0$, $\lambda T_{y,w} = \lambda_{so}T_{n,w}$, where λ_{so} is the constant thermal conductivity coefficient of the solid. In dimensionless form, the last equation may be written as $\lambda t_{y,w} = [t_w - 1]/\Phi x$ in the first case and $\lambda t_{y,w} = -(xt_x)_{x,w}/\Phi$ in the second case, where the coupling parameter Φ is $\Phi = Re^{1/2}\lambda_2\alpha/\lambda_{so}$ and $\Phi = Re^{1/2}\lambda_2/(\lambda_{so}\alpha)$, respectively. To take into account the influence of variability of the fluid properties on the flow, it is convenient to introduce the Stewartson-Dorodnitzin transformation by using the new independent variables:

$$\xi = x, \qquad \eta = \int_0^y \rho \, dy, \qquad V = \rho v + u\eta_x$$

In the case of $\rho\mu$ const, the continuity and momentum equations may be solved independently of the energy equation, and the streamfunction Ψ is obtained in similarity form in terms of the variable $z = \eta/\xi^{1/2}$. Instead we assume the following dependence of μ and λ on absolute temperature T:

$$\mu = \sum_{i=1}^{n} \alpha_i T_i, \qquad \lambda = \sum_{i=1}^{n} \beta_i T_i$$

so that, in dimensionless form, products $\rho\mu$ and $\rho\lambda(\rho = 1/T$ for a perfect gas) are expressed by means of polynomials in terms of t:

$$\rho\mu = 1 + \sum_{i=1}^{n-1} a_i t_i, \qquad \rho\lambda = 1 + \sum_{i=1}^{n-1} b_i t_i$$

If we consider $n = 2$, it is $\rho\mu = 1 + a_1 t$, $\rho\lambda = 1 + b_1 t$. In terms of these variables, one has $(1 + b_1 t_w)t_{z,w} = [t_w - 1]/\Phi\xi^{1/2}$, $(1 + b_1 t_w)t_{z,w} = -(\xi t_\xi)_{\xi,w}\xi^{1/2}/\Phi$ for each case, respectively.

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Method of Solution

In the first case, owing to the coupling condition, the problem does not admit similarity solutions; in fact, the solution has a different character for small and high values of ξ ($\xi = x$). For small values of ξ, the solution tends to that obtained with the isothermal condition at the interface, and it is convenient to assume $m_1 = \Phi\xi^{1/2}$ and z as independent variables. In this case, it is possible to expand the unknowns in a Maclaurin series with respect to m_1. This expansion, whose leading term represents the solution with the isothermal condition at the interface, has a finite radius of convergence and does not allow us to describe the entire thermofluiddynamic field. However, it is possible, using the Padé approximant techniques, to not only calculate the radius of convergence of the initial expansion, but also to obtain a representation valid in the whole field. For high values of ξ, the solution tends to that obtained with the adiabatic condition at the interface, and it is convenient to assume $m_2 = 1/m_1$ and z as independent variables. In this case, it is not possible to expand the unknowns in a Maclaurin series with respect to m_2 because, although no problem is encountered in calculating the first two coefficients of the expansion, the third one cannot be calculated for the presence of eigensolutions. Nevertheless, since terms of this kind do not appear below order 2, they can be neglected in the proposed procedure. Each term of the two expansions is obtained by solving numerically a system of ordinary differential equations obtained in the usual way applying the Cauchy's rule for multiplication of power series. In particular, in the following we assume, for the sake of simplicity, $n = 2$ in the expressions of $\rho\mu$ and $\rho\lambda$. To determine the equations and the boundary conditions that allow us to calculate the terms of the expansions for small (initial expansion) and high (asymptotic expansion) values of ξ, we shall follow a unified procedure, assuming $z = \eta/\xi^{1/2}$ and $m = (\Phi\xi^{1/2})^\delta$ as independent variables, where $\delta = 1$ for the initial expansion ($m = m_1$), and $\delta = -1$ for the asymptotic expansion ($m = m_2$). Then, letting $\Psi = \xi^{1/2}f(\xi,z)$ ($u = \Psi_\eta$ and $V = -\Psi_\xi$) and $t = h(\xi,z)$ and assuming z and m as independent variables, we expand the functions f and h in a Maclaurin series with respect to m by putting

$$f = \sum_{i=0} m^i f_i(z),\ h = \sum_{i=0} m^i h_i(z)$$

The expansion for small values of ξ (where $m = \Phi\xi^{1/2}$) is regular and its radius of convergence can be calculated by means of Padé approximant techniques. The idea of Padé summation is to replace a power series $\Sigma c_n s^n$ by a sequence of rational functions of the form

$$P_M^N(s) = \sum_{n=0}^{N} C_n s^n / \sum_{n=0}^{M} D_n s^n$$

In this way it is possible to obtain a rapid convergence by using only a few terms of the original Taylor series, but above all the utility of Padé approximants lies in the fact that they work well also when the Taylor series does not converge, giving a representation valid in the entire thermofluiddynamic field. On the contrary, the expansion for high values of ξ(where $m = 1/(\Phi\xi^{1/2})$) is not regular and only the first two terms of this expansion may be calculated without any difficulties. It is necessary, therefore, to modify the form of the asymptotic expansion and to give initial conditions at $\xi = \xi_0 > 0$. However, it is sufficient to consider only the first two terms of the asymptotic expansion, because the Padé approximant techniques will permit us to obtain a representation valid in the entire thermofluiddynamic field.

When the axis is under adiabatic conditions (the second case), the thermal boundary condition at the solid-fluid interface is $(1 + b_1 t_w)t_{z,w} = -(\xi t_\xi)_{\xi,w}\xi^{1/2}/\Phi$. Then, the boundary-layer equations may be solved assuming t and f (such that $\Psi = \xi^{1/2}f$) to be functions of the similarity variable z only.

Results and Discussion

The previous analysis has been applied to the case of air for the Mach number M_1 upstream of the shock wave with values of 3 and 6, and the value 300 K for T_1; in addition, we have assumed $\alpha = 5$ deg and $T_a = 1000$ K. For $M_1 = 3$, $T_1 = 270$ K and $\alpha = 5$ deg, one obtains, from the standard wave theory, $M_2 = 2.75$ and $T_2 = 301$ K, whereas for $M_1 = 6$ and the same values of T_1 and α one has $M_2 = 5.32$ and $T_2 = 332$ K. As noted in the previous section, Padé approximants technique has been used for the representation of results in the case of the solution with the isothermal boundary condition, since the Padé representation is valid even when the Maclaurin original expansion does not converge. For $M_1 = 3$, the approximation $\rho\mu = \rho\lambda = 1$ provides very good results, whereas for $M_1 = 6$, the difference is about 10%. At $m_1 = 0$, $t_w = 1$ and the solution is close to that obtained with the isothermal boundary condition, whereas for high values of m_1 the solution tends to be that obtained with the adiabatic condition. It must be noted that, since the asymptotic solution is obtained with the adiabatic condition, the curves of $Nu_x/Re_x^{1/2}$ tend to zero for $m_1 \to \infty$ ($x \to \infty$). If we assume $a_1 = b_1 = 0$, c_f does not depend on the variable m_1; in fact, in this case the velocity field may be solved independently of the thermal field in terms of the similarity variable z; for the evaluatation of c_f the assumption $\rho\mu = \rho\lambda = 1$ is not satisfactory even for $M_1 = 3$. The difference with respect to the case $a_1 = b_1 = 0$ is about 9% for $M_1 = 3$ and about 30% for $M_1 = 6$. The solution with the adiabatic boundary condition has been obtained in a similarity form in terms of the variable z. The initial values $t(0)$ and $f''(0)$ depend on the values of a_1 and b_1. In order to analyze the influence of the variable properties of the fluid on flow, the curves of $t_w = t(0)$ and c_f vs a_1 have been plotted for the values $M_1 = 3$ and $b_1 = -0.4, 0, 0.4$, and can be found in the backup paper. These figures show that both t_w and c_f increase almost linearly with a_1.

References

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